3D Flexible Multibody Thin Beam simulation in Modelica with the Finite Element Method

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Abstract

This paper presents the development, simulation and validation of 3 dimensional flexible beam models using *Modelica*. The models are based on finite element method (FEM) application, following mathematical calculations proposed by Shabana [2], and are 3D extensions to the 2D model developed by F. Schiavo [3]. The element formulation is independent of applied boundary conditions, making the element suitable for any 3D multibody simulation.
All models use standard connectors defined in the *Modelica* multibody library, thereby guaranteeing full compatibility with library components. Mathematical modelling details are fully analyzed, indicating motion equation development. Models also feature a graphical interface, and visualization of simulation outcomes using the same 3D environment as the multibody library, providing the user with immediate visual feedback. Finally, models are analyzed and validated by means of selected simulation experiments, with reference to theoretical predictions and comparison with results obtained with commercial FEM code.

Keywords: flexible beam; Multibody; FEM; finite element formulation

1 Introduction

Whilst 2D flexible model development is widely covered by literature [2], its application to 3 Dimensional models normally only goes as far as the initial stages of the mathematical development. This paper presents the application of 2D model redevelopment and implementation [3] to 3D models using *Modelica*. At times shall descriptions made by Schiavo [3] are used.

With the aim of creating a reusable parametric element, is has been considered only the modelling of one particular scenario: a beam containing two connection points, one at each end.

In this paper the linear elasticity theory for thin beam modelling are considered, ignoring shear deformation effects and assuming uniform cross sectional properties throughout the element length. Cross sectional dimensions compared to element length rigid configuration deflection are also assumed to be small.

Taking into account the object-oriented capability of the *Modelica* language, some parametric features have been implemented in the model, for example: cross sectional beam shape (rectangular or cylindrical), hollow or full section, and mesh length density (from 1 to N elements). This provides a parametric mesh of the element enabling computation of points along the element length and precision-computational cost control.

The implementation of other shape sections can be easily carried out, but are not included in the model in order to simplify the user interface menu.

It has been also developed a model that for importing mass and stiffness matrix values from condensed FE models to 12 degrees of freedom (dof).

2 Degrees of Freedom (dof)

Consider a generic multibody system (Figure 1). The position, in body coordinates, of a point in a specific deformable body is expressed as follows:

$$u = u_0 + u_f \qquad (1)$$

where u_0 is the “undeformed” (i.e., rigid) position vector and u_f is the deformation contribution to position (i.e., the deformation field).

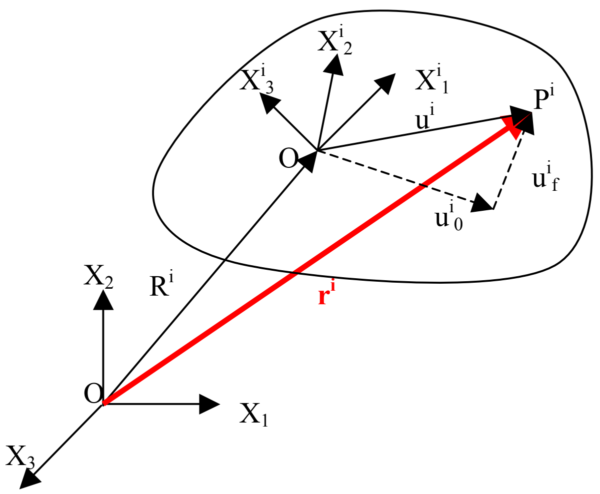

Figure 1: Flexible body reference systems

The mathematical description of a body's generic deformation requires that the deformation field belong to an infinite dimensional functional space, requiring, in turn, an infinite number of deformation degrees of freedom.

In this paper, the deformation field is described by an approximation to the functional basis space it belongs to, assuming that such space has a finite dimension, say M, so that vector u_f can be expressed by the following finite dimensional product:

$$u_f = Sq_f \tag{2}$$

where S is the [3×M] shape function matrix (i.e., a matrix of functions defined over the body domain and used as a basis to describe the deformation field of the body itself) and q_f is the M-dimensional vector of deformation degrees of freedom.

The position of a point in a deformable body can then be expressed in world reference as follows:

$$r = R + Au = R + A\left(u_0 + Sq_f\right) = R + Au_0 + ASq_f \tag{3}$$

where R is the vector identifying the origin of the body local reference system and A is the rotation matrix for the body reference system.

The representation of a generic deformable body in world reference requires then 6+M d.o.f. (i.e., 6 corresponding to rigid displacements and rotations and M to deformation fields):

$$q = \left[q_r \; q_f\right]^T = \left[R \; \theta \; q_f\right]^T \tag{4}$$

where R and θ represents unreformed body position and orientation angles and q_f is a vector containing flexible degrees of freedom.

3 Motion Equations

The equations are solved using a classical Lagrangian approach. The equation for flexible element motion, in body axes, can be expressed as [2],[3] (a general demonstration to motion equations in [2] and in more detail in [3]):

$$\begin{bmatrix} m_{RR} & \tilde{S}_t^T & \bar{S} \\ & \bar{I}_{\theta\theta} & \bar{I}_{\theta f} \\ & & m_{ff} \end{bmatrix} \cdot \begin{bmatrix} \ddot{R} \\ \bar{\alpha} \\ \ddot{q}_f \end{bmatrix} = \\ = \begin{bmatrix} O_3 \\ O_3 \\ -K_{ff} q_f \end{bmatrix} + \begin{bmatrix} Q_v^R \\ Q_v^\theta \\ Q_v^f \end{bmatrix} + \begin{bmatrix} \bar{Q}_e^R \\ \bar{Q}_e^\theta \\ \bar{Q}_e^f \end{bmatrix} \tag{5}$$

Equations are valid for a general deformable body, though many of the quantities involved (e.g., the Kff matrix) depend on specific body characteristics such as the shape and material properties, but do not depend on element deformation.

The mass matrix obtained using the formulation of [3], takes the following form:

$$M^i = \begin{bmatrix} M^i_{RR} & M^i_{R\theta} & M^i_{Rf} \\ M^{i\,T}_{\theta R} & M^i_{\theta\theta} & M^i_{\theta f} \\ M^{i\,T}_{fR} & M^{i\,T}_{f\theta} & M^i_{ff} \end{bmatrix} \tag{6}$$

Where mass matrix components (6) are calculated in the following manner, assuming that the body is a 3D elastic continuum, with constant cross-sectional properties, isotropic material behaviour and is perfectly elastic.

$$M^i_{RR} = \int_{V^i} \rho^i I dV^i \tag{7}$$

$$M^i_{R\theta} = M^{i\,T}_{\theta R} = -A^i \left[\int_{V^i} \rho^i \tilde{\bar{u}}^i dV^i \right] \bar{G}^i \tag{8}$$

$$M^i_{Rf} = A^i \int_{V^i} \rho^i S^i dV^i = A^i \bar{S}^i \tag{9}$$

$$M_{\theta\theta}^{i} = \int_{V^i} \rho^i \left(A^i \tilde{u}^i \bar{G}^i\right)^T \left(A^i \tilde{u}^i \bar{G}^i\right) dV^i \tag{10}$$

$$M_{\theta f}^{i} = \bar{G}^{i^T} \int_{V^i} \rho^i \tilde{u}^i S^i dV^i \tag{11}$$

$$M_{ff}^{i} = \int_{V^i} \rho^i S^{i^T} S^i dV^i = \bar{S}_{11}^i + \bar{S}_{22}^i + \bar{S}_{33}^i \tag{12}$$

Where the 3D shape matrix is:

$$S^{ij^T} = \begin{bmatrix} 1-\xi & 0 & 0 \\ 6\left[\xi-(\xi)^2\right]\eta & 1-3(\xi)^2+2(\xi)^3 & 0 \\ 6\left[\xi-(\xi)^2\right] & 0 & 1-3(\xi)^2+2(\xi)^3 \\ 0 & -(1-\xi)l\zeta & -(1-\xi)l\eta \\ \left[1-4\xi+3(\xi)^2\right]l\zeta & 0 & \left[-\xi+2(\xi)^2-(\xi)^3\right]l \\ \left[1-4\xi+3(\xi)^2\right]l\eta & \left[\xi-2(\xi)^2+(\xi)^3\right]l & 0 \\ \xi & 0 & 0 \\ 6\left[-\xi+(\xi)^2\right]\eta & 3(\xi)^2-2(\xi)^3 & 0 \\ 6\left[-\xi+(\xi)^2\right]\zeta & 0 & 3(\xi)^2-2(\xi)^3 \\ 0 & -l\xi\zeta & l\xi\eta \\ \left[-2\xi+3(\xi)^2\right]l\zeta & 0 & \left[(\xi)^2-(\xi)^3\right] \\ \left[-2\xi+3(\xi)^2\right]l\eta & \left[-(\xi)^2+(\xi)^3\right]l & 0 \end{bmatrix}^{ij} \tag{13}$$

To be able to perform the calculation at the right beam extremity (point B, Figure 2) it has been supposed that it is calculated exactly at the average point of the element in order to simplify matrices that intervene in the mass matrix calculation.

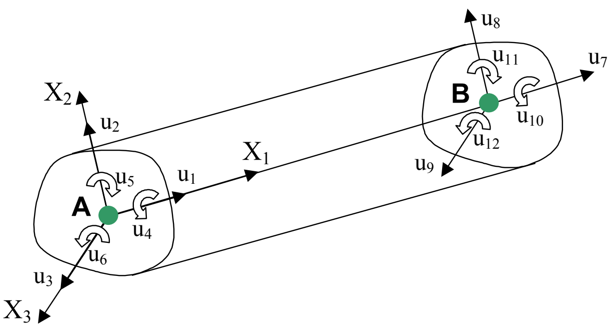

Figure 2: Beam extremities

By means of this assumption the general matrices are simplified applying the following assumption:

$$\xi^{ij} = \frac{x_1^{ij}}{l^{ij}} = 1 \tag{14}$$

$$\eta^{ij} = \frac{x_2^{ij}}{l^{ij}} = 0 \tag{15}$$

$$\zeta^{ij} = \frac{x_3^{ij}}{l^{ij}} = 0 \tag{16}$$

In the end, by using these assumptions, the beam is simplified to a line due to the fact that the length of the beam and the height and width they are not taken into account.

It also implies that the following terms vanish in the mass matrix formulation (see appendix B)

$$Q_\eta^{ij} = \left[\int_a \rho\eta da\right]^{ij} = 0 \tag{17}$$

$$Q_\zeta^{ij} = \left[\int_a \rho\zeta da\right]^{ij} = 0 \tag{18}$$

$$I_\zeta^{ij} = \left[\int_a \rho(\eta)^2 da\right]^{ij} = 0 \tag{19}$$

$$I_\eta^{ij} = \left[\int_a \rho(\zeta)^2 da\right]^{ij} = 0 \tag{20}$$

$$I_{\eta\zeta}^{ij} = \left[\int_a \rho\eta\zeta da\right]^{ij} = 0 \tag{21}$$

Following the procedure described in [3] and the matrix in appendix B, one can verify that the integrals that appear in the expression of S_{kl}^{ij} are given by:

$$\left[\int_V \rho S_1^T S_1 dV\right]^{ij} = \\ = \begin{bmatrix} \frac{m}{3} & 0 & 0 & 0 & 0 & 0 & \frac{m}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{m}{6} & 0 & 0 & 0 & 0 & 0 & \frac{m}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{ij} \tag{22}$$

$$\left[\int_V \rho S_2^T S_2 dV\right]^{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{13m}{35} & 0 & 0 & 0 & \frac{11ml}{210} & 0 & \frac{9m}{70} & 0 & 0 & 0 & -\frac{13ml}{420} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11ml}{210} & 0 & 0 & 0 & \frac{m(l)^2}{105} & 0 & \frac{13ml}{420} & 0 & 0 & 0 & -\frac{m(l)^2}{140} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{9m}{70} & 0 & 0 & 0 & \frac{13ml}{420} & 0 & \frac{13m}{35} & 0 & 0 & 0 & -\frac{11ml}{210} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{13ml}{420} & 0 & 0 & 0 & -\frac{m(l)^2}{140} & 0 & -\frac{11ml}{210} & 0 & 0 & 0 & \frac{m(l)^2}{105} \end{bmatrix}^{ij} \quad (23)$$

$$\left[\int_V \rho S_3^T S_3 dV\right]^{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{13m}{55} & 0 & -\frac{11ml}{210} & 0 & 0 & 0 & \frac{9m}{70} & 0 & \frac{13ml}{420} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{11ml}{210} & 0 & \frac{m(l)^2}{105} & 0 & 0 & 0 & -\frac{13ml}{420} & 0 & -\frac{m(l)^2}{140} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{9m}{70} & 0 & -\frac{13ml}{420} & 0 & 0 & 0 & \frac{13m}{35} & 0 & \frac{11ml}{210} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{13m}{420} & 0 & -\frac{m(l)^2}{140} & 0 & 0 & 0 & \frac{11ml}{210} & 0 & \frac{m(l)^2}{105} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{ij} \quad (24)$$

$$\left[\int_V \rho S_1^T S_3 dV\right]^{ij} = \begin{bmatrix} 0 & 0 & \frac{7m}{20} & 0 & -\frac{ml}{20} & 0 & 0 & 0 & \frac{3m}{20} & 0 & -\frac{ml}{30} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3m}{20} & 0 & -\frac{ml}{30} & 0 & 0 & 0 & \frac{7m}{20} & 0 & \frac{ml}{20} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{ij} \quad (25)$$

$$\left[\int_V \rho S_2^T S_3 dV\right]^{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{13m}{35} & 0 & -\frac{11ml}{210} & 0 & 0 & 0 & \frac{9m}{70} & 0 & \frac{13ml}{420} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{11ml}{210} & 0 & -\frac{m(l)^2}{105} & 0 & 0 & 0 & \frac{13ml}{420} & 0 & \frac{m(l)^2}{140} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{9m}{70} & 0 & -\frac{13ml}{420} & 0 & 0 & 0 & \frac{13m}{35} & 0 & \frac{11ml}{210} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{13ml}{420} & 0 & \frac{m(l)^2}{140} & 0 & 0 & 0 & -\frac{11ml}{210} & 0 & -\frac{m(l)^2}{105} & 0 \end{bmatrix}^{ij} \quad (26)$$

$$\left[\int_V \rho S_1^T S_2 dV\right]^{ij} = \begin{bmatrix} 0 & \frac{7m}{20} & 0 & 0 & 0 & \frac{ml}{20} & 0 & \frac{3m}{20} & 0 & 0 & 0 & -\frac{ml}{30} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3m}{20} & 0 & 0 & 0 & \frac{ml}{30} & 0 & \frac{7m}{20} & 0 & 0 & 0 & -\frac{ml}{20} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{ij} \quad (27)$$

It can be seen that the mass matrix depends on element deformation, and needs to be re-calculated for each simulation instance.

The speed quadratic vector is a non-linear function of the widespread system of coordinates and speeds and includes the Coriolis effect and the effect of centrifugal forces.

$$\begin{aligned} \left(Q_v^i\right)_R &= -A^i\left[\left(\overline{w}^i\right)^2 S_t^i + 2\overline{w}^j \overline{S}^j \dot{q}_f^i\right] \\ \left(Q_v^i\right)_\theta &= -2\overline{G}^{iT} \overline{I}_{\theta\theta}^i \overline{w}^i - 2\overline{G}^{iT} \overline{I}_{\theta f}^i \dot{q}_f^i - \overline{G}^{iT} \overline{I}_{\theta\theta}^i \overline{w}^i \\ \left(Q_v^i\right)_f &= -\int_{V^i} \rho^i \left\{S^{iT}\left[\left(\overline{w}^i\right)^2 \overline{u}^i + 2\overline{w}^i \overline{u}_f^i\right]\right\} dV^i \end{aligned} \quad (28)$$

3.1 The element point of view

The finite element method is based upon a discretization of the beam into N elements. It is then possible to define the local dimensionless *abscissa* as $\xi = x/l$, where x is the longitudinal local coordinate and l is the element length.

For a single element, the generic equations of motion (5) can be expanded as follows:

$$\bar{u}_{f,el} = \begin{bmatrix} \bar{u}_{f1,el} \\ \bar{u}_{f2,el} \\ \bar{u}_{f3,el} \end{bmatrix} = S_{el} q_{f,el} \begin{cases} q_{f,el}{}^{T} = \begin{bmatrix} q_{f1,el} \\ q_{f2,el} \\ q_{f3,el} \\ q_{f4,el} \\ q_{f5,el} \\ q_{f6,el} \\ q_{f7,el} \\ q_{f8,el} \\ q_{f9,el} \\ q_{f10,el} \\ q_{f11,el} \\ q_{f12,el} \end{bmatrix} \\ S_{el} = \begin{bmatrix} S_{el1} \\ S_{el2} \\ S_{el3} \end{bmatrix} \end{cases} \tag{29}$$

where the subscript *el* is used to refer the quantities of a single element.

Figure 2 depicts the element coordinate systems associated with the deformation degrees of freedom: $q_{f1,el}$ and $q_{f7,el}$ are associated with axial compression, $q_{f2,el}$ and $q_{f8,el}$ with transversal displacement, $q_{f3,el}$ and $q_{f9,el}$ with Z axis displacement, $q_{f4,el}$ and $q_{f10,el}$ with beam extremities rotation around the X axis, $q_{f5,el}$ and $q_{f11,el}$ with beam extremities rotation around the Y axis and $q_{f6,el}$ and $q_{f12,el}$ with beam extremity rotation around the Z axis.

3.2 Finite Element Method Equation Assembly

The motion equations for the entire beam can be obtained by assembling the motion equations for beam elements as defined in the previous subsection. The body reference system will be the local reference system located at the root of the first element, so that the rigid degrees of freedom, common to all the elements, will refer to such a coordinate system.
Let then m and L be the mass and length of the entire beam, and N the number of elements used, so that $l=L/N$. With $\hat{X}$ indicating the reference system unit vector along the beam axis, the expression of the generic position u_j of a point of element j can be expressed as:

$$\bar{u}_j = \bar{u}_{0j} + S_{el} B_j q_f = \left[\xi_j l + (j-1)l\right]\hat{X} + S_{el} B_j q_f \tag{30}$$

where u_{0j} is the position of the root of the j^{th} element, S_{el} is the shape functions matrix defined by (31), B_j is the so-called *connectivity matrix* and q_f is a vector containing the deformation degrees of freedom for the whole beam. Matrices B_j have the following form:

$$B_j = \left[O_{6,3(j-1)} \quad I_6 \quad O_{6,3(N-j)}\right], \forall\, j = 1, \ldots, N \tag{31}$$

The connectivity matrices are used to relate vector q_f, which contains the deformation degrees of freedom for the entire beam, to the corresponding j^{th} element, according to the expression:

$$q_{f,el_j} = B_j q_f \tag{32}$$

4 Modelica Implementation

The finite element model formulation has been implemented using the *Modelica* language, creating thus a new component, called *FlexBeamFem3D* (Figure 4). The component interfaces are two standard mechanical flanges from the new *MultiBody* library [4]. The choice of connector makes the component fully compatible with the *Modelica Multibody* library, so that it is possible to directly connect the flexible beam component to the predefined models, such as mechanical constraints (revolute joints, prismatic joints, etc.), parts (3D rigid bodies) and force elements (springs, dampers, forces, torques).

Figure 4: Component icon

The models also have a graphical interface, with a visualization of simulation outcomes within the same 3D environment used in the *Multibody* library, providing the user with immediate visual feedback.

Mathematical modelling details are partially indicated below in Modelica characteristic language:

```
parameter SI.Density rho=7800 "Material Volume Density";
parameter SI.Length L=0.2 "Beam Length";
parameter SI.Height a=0.005 "Height of section";
parameter SI.Breadth b=0.02 "Breath of section";
parameter SI.ModulusOfElasticity E=210e9 "Material Youngs modulus";
parameter SI.ShearModulus G=8.077e10 "Material Shear modulus";
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parameter SI.SecondMomentOfArea J1=7.025e-10 "Cross sectional inertia 1";
parameter SI.SecondMomentOfArea J2=3.33333e-9 "Cross sectional inertia 2";
parameter SI.SecondMomentOfArea J3=2.08333e-10 "Cross sectional inertia 3";
parameter Real Alpha=1e-3 "Rayleigh structural damping proportional to mass [sec^-1]";
parameter Real Beta=5e-5 "Rayleigh structural damping proportional to stiffness [sec]";
parameter Integer N(min=2) = 2 "Number of Elements";

Dynamic equations for the 3D flexible beam model are as follows:

*[m*identity(3), transpose(StbarCross), Sbar]*[aa - g_0; za; ddqf] = QvR +matrix(fa + fb_a);*

[StbarCross, Ithth_bar, Ithf_bar][aa - g_0; za; ddqf] = QvAlpha + matrix(ta + tb_a + cross(({L,0,0} + S1*B[N, :, :]*qf), fb_a));*

[transpose(Sbar), transpose(Ithf_bar), mff][aa - g_0; za; ddqf] = Qvf + Qef - matrix(Kff*qf) - matrix((Alpha*mff + Beta*Kff)*dqf);*

Using the degrees of freedom q_f and the derivate of degrees of freedom dq_f, information can be passed from *Frame B* to *Frame A* as indicated in the following equations:

FrameB.R=Modelica.Mechanics.MultiBody.Frames.absoluteRotation(FrameA.R, R_rel);

*R_rel=Modelica.Mechanics.MultiBody.Frames.axesRotations({1,2,3},{qf[(6*N)-2],qf[(6*N)-1],qf[6*N]},{dqf[(6*N)-2],dqf[(6*N)-1],dqf[6*N]});*

5 Simulations

The different flexible beam models have been validated by several simulation analyses performed in the *Dymola* simulation environment [1] and compared with a general purpose commercial FEM code (ANSYS).

5.1 First example (Static)

As a preliminary result, a 3D simulation has been performed applying movement to the free extremity of a cantilever beam (Figure 5).

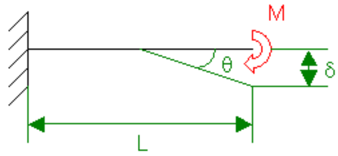

Figure 5:Beam 3D

Beam displacement and rotation analytical form calculation follows:

$$\delta = \frac{M\,L^2}{2EI} = \frac{\frac{2\pi EI}{L}L^2}{2EI} = \pi L \quad (33)$$

$$\theta = \frac{M\,L}{EI} = \frac{\frac{2\pi EI}{L}L}{EI} = 2\pi \quad (34)$$

Variables used in the simulation are expressed in the following table:

Variable	Value
Rho (kg/m^3)	7800
L (m)	1
a (m)	0.02
b (m)	0.06285
E (Pa)	210e9
G (Pa)	8.077e10
J1(m^4)	1.33242e-7
J2(m^4)	4.138e-7
J3(m^4)	4.19e-8
Alpha	1e-3
Beta	5e-5
N	2
Applied moment M (N.m)	2*π*E*I/L

The simulation has been performed by connecting 10 elements in series (Figure 6), to overcome the assumption of small displacements in the internal development of the element.

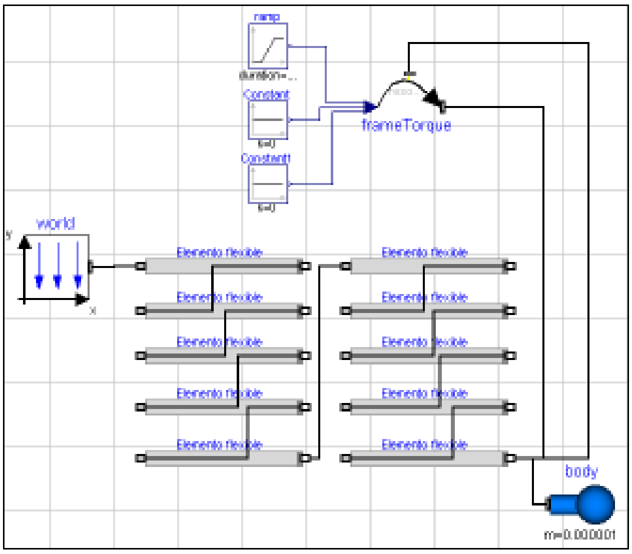

Figure 6: Modeling in Dymola

Figure 7 shows the results obtained where the 3D position of the beam's extremities is plotted at the end of the simulation. As expected, X and Y positions describe a circle since as this was verified in the theoretical analysis. The same results are obtained, irrespective of the direction of movement applied.

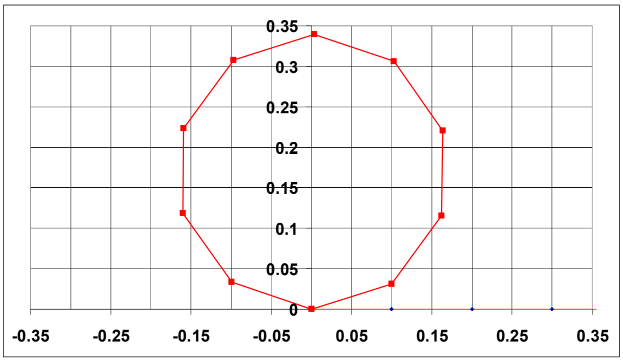

Figure 7: Results obtained

By means of this simulation correct element response to static flexion has been obtained.

5.2 Second example (Dynamic)

In this example (Figure 8) the dynamical operation of a flexible pendulum articulated to one of its ends has been analysed. The initial position of the flexible element is horizontal (X direction) and gravity is applied in the vertical direction (Z direction).

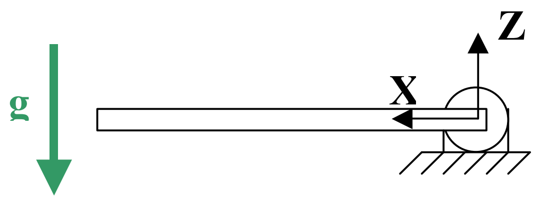

Figure 8: Articulated pendulum

Figure 9 shows the comparison of the results obtained with Dymola, ANSYS code and considering a rigid element. The mechanical properties of the beam are the same as in the previous example except L being 0.2. This simulation can be performed with one or more elements because the large displacement of the end is mainly governed by the revolute joint, the flexibility acting only in the longitudinal direction of the beam.

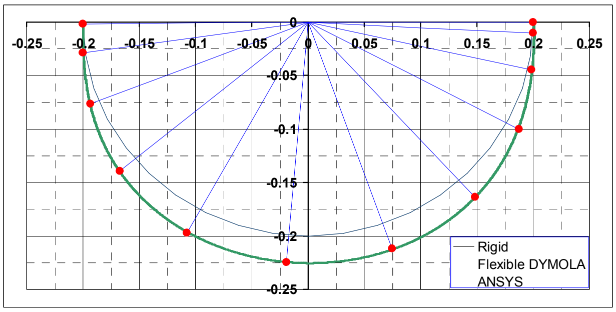

Figure 9: Results obtained with different programs

Between the rigid model and the flexible model there is a lack of coordination in frequency where the rigid element is advanced with regard to the flexible model. The results concur with those obtained in ANSYS.

6 Generalization and model versions

In the general and parametric 3D model created, modifications have been performed in order to introduce mass and stiffness (matrices calculated for example using ANSYS). In this case the matrix of mass becomes constant along the simulation as well as it happens with the matrix of stiffness and besides Coriolis's terms though they appear are not updated.

By means of this simplification the computational cost decreases because the reduction in the calculation of the equations of movement. It must be pointed out that a calculation error is made, however this error could be fully undertaken.

7 Conclusions

To date, there has been no evidence of a Dymola model that can simulate flexible mechanisms in 3D and therefore the importance of the model created, as its correct operation has been verified, both statically and dynamically.

This development in flexible element simulation, based on *Modelica* programming language, enables the simulation of flexible 3D mechanisms and the integration of these model into other disciplines, for example, control tasks.

The only drawback to the implemented model is the large number of equations required to solve the problem, this slows down simulation and requires powerful simulation equipment. Although, compared to other simulation programs, the 3D flexible model created *Dymola* is about 100 times faster than the same model simulated in ANSYS.

To overcome this limitation a simplified model has been implemented that considers a fixed mass matrix and a single element (no discretisation), that speeds up simulation, and is suitable for preliminary modeling steps.

The developed model can be easily implemented for any type of constant shape along its length.

Future work will include non-constant shape beams and the development of a model based on the assumed modes theory [3] considering free boundaries.

This flexible model will be used for the development of a wearable robotics where the mechanism mass is key.

A Structural Stiffness Matrix

$$K_{ff}^{i}=\begin{bmatrix}
\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{12EI_3}{(l)^3} & 0 & 0 & 0 & \frac{6EI_3}{(l)^2} & 0 & -\frac{12EI_3}{(l)^3} & 0 & 0 & 0 & \frac{6EI_3}{(l)^2}\\
0 & 0 & \frac{12EI_2}{(l)^3} & 0 & -\frac{6EI_2}{(l)^2} & 0 & 0 & 0 & -\frac{12EI_2}{(l)^3} & 0 & -\frac{6EI_2}{(l)^2} & 0\\
0 & 0 & 0 & \frac{GJ_1}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ_1}{l} & 0 & 0\\
0 & 0 & -\frac{6EI_2}{(l)^2} & 0 & \frac{4EI_2}{l} & 0 & 0 & 0 & \frac{6EI_2}{(l)^2} & 0 & \frac{2EI_2}{l} & 0\\
0 & \frac{6EI_3}{(l)^2} & 0 & 0 & 0 & \frac{4EI_3}{l} & 0 & -\frac{6EI_3}{(l)^2} & 0 & 0 & 0 & \frac{2EI_3}{l}\\
-\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{12EI_3}{(l)^3} & 0 & 0 & 0 & -\frac{6EI_3}{(l)^2} & 0 & \frac{12EI_3}{(l)^3} & 0 & 0 & 0 & -\frac{6EI_3}{(l)^2}\\
0 & 0 & -\frac{12EI_2}{(l)^3} & 0 & \frac{6EI_2}{(l)^2} & 0 & 0 & 0 & \frac{12EI_2}{(l)^3} & 0 & \frac{6EI_2}{(l)^2} & 0\\
0 & 0 & 0 & -\frac{GJ_1}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ_1}{l} & 0 & 0\\
0 & 0 & -\frac{6EI_2}{(l)^2} & 0 & \frac{2EI_2}{l} & 0 & 0 & 0 & \frac{6EI_2}{(l)^2} & 0 & \frac{4EI_2}{l} & 0\\
0 & \frac{6EI_3}{(l)^2} & 0 & 0 & 0 & \frac{2EI_3}{l} & 0 & -\frac{6EI_3}{(l)^2} & 0 & 0 & 0 & \frac{4EI_3}{l}
\end{bmatrix}$$

B Mass Matrix Components

$$\left[\int_V \rho S_1^T S_1 dV\right]^{ij} =$$

$$=\begin{bmatrix}
\frac{m}{3} &&&&&&&&&&&\\
\frac{Q_\eta l}{2} & \frac{6I_\zeta l}{5} &&&&&&&&&&\\
\frac{Q_\zeta l}{2} & \frac{6I_{\eta\zeta} l}{5} & \frac{6I_\eta l}{5} &&&&&&&&&\\
0 & 0 & 0 & 0 &&&&& \textit{symmetric} &&&\\
\frac{Q_\zeta (l)^2}{12} & -\frac{I_{\eta\zeta}(l)^2}{10} & -\frac{I_\eta (l)^2}{10} & 0 & \frac{2I_\eta (l)^3}{15} &&&&&&&\\
-\frac{Q_\eta (l)^2}{12} & \frac{I_\zeta (l)^2}{10} & \frac{I_{\zeta\eta}(l)^2}{10} & 0 & -\frac{2I_{\eta\zeta}(l)^3}{15} & \frac{2I_\zeta (l)^3}{15} &&&&&&\\
\frac{m}{6} & \frac{lQ_n}{2} & \frac{Q_\zeta l}{2} & 0 & -\frac{Q_\eta (l)^2}{12} & \frac{Q_\eta (l)^2}{12} & \frac{m}{3} &&&&&\\
-\frac{Q_\eta l}{2} & -\frac{6I_\zeta l}{5} & -\frac{6I_{\eta\zeta} l}{5} & 0 & \frac{I_{\eta\zeta}(l)^2}{10} & -\frac{I_\zeta (l)^2}{10} & -\frac{Q_\eta l}{2} & \frac{6I_\zeta l}{5} &&&&\\
-\frac{Q_\zeta l}{2} & -\frac{6I_{\eta\zeta} l}{5} & -\frac{6I_\eta l}{5} & 0 & \frac{I_\eta (l)^2}{10} & -\frac{I_{\eta\zeta}(l)^2}{10} & -\frac{Q_\zeta l}{2} & \frac{6I_{\eta\zeta} l}{5} & \frac{6I_\eta l}{5} &&&\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&\\
-\frac{Q_\zeta (l)^2}{12} & -\frac{I_{\eta\zeta}(l)^2}{10} & -\frac{I_\eta (l)^2}{10} & 0 & -\frac{I_\eta (l)^3}{30} & \frac{I_{\eta\zeta}(l)^3}{30} & \frac{Q_\zeta (l)^2}{12} & \frac{I_{\eta\zeta}(l)^2}{10} & \frac{I_\eta (l)^2}{10} & 0 & \frac{2I_\eta (l)^3}{15} &\\
-\frac{Q_\eta (l)^2}{12} & \frac{I_\zeta (l)^2}{10} & \frac{I_{\zeta\eta}(l)^2}{10} & 0 & \frac{I_{\eta\zeta}(l)^3}{30} & -\frac{I_\zeta (l)^3}{30} & -\frac{Q_\eta (l)^2}{12} & -\frac{I_\zeta (l)^2}{10} & -\frac{I_{\zeta\eta}(l)^2}{10} & 0 & -\frac{2I_{\eta\zeta}(l)^3}{15} & \frac{2I_\zeta (l)^3}{15}
\end{bmatrix}^{ij}$$

$$\left[\int_V \rho S_2^T S_2 dV\right]^{ij} =$$

$$=\begin{bmatrix}
0 &&&&&&&&&&&\\
0 & \frac{13m}{35} &&&&&&&&&&\\
0 & 0 & 0 &&&&&&&&&\\
0 & -\frac{7(l)^2}{20}Q_\zeta & 0 & \frac{(l)^3}{3}I_\eta &&&&& \textit{symmetric} &&&\\
0 & 0 & 0 & 0 & 0 &&&&&&&\\
0 & \frac{11ml}{210} & 0 & -\frac{(l)^3}{20}Q_\zeta & 0 & \frac{m(l)^2}{105} &&&&&&\\
0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&\\
0 & \frac{9m}{70} & 0 & -\frac{3(l)^2}{20}Q_\zeta & 0 & \frac{13ml}{420} & 0 & \frac{13m}{35} &&&&\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&\\
0 & -\frac{3(l)^2}{20}Q_\zeta & 0 & \frac{(l)^3}{6}I_\eta & 0 & -\frac{(l)^3}{30}Q_\zeta & 0 & -\frac{7(l)^2}{20}Q_\zeta & 0 & \frac{(l)^3}{3}I_\eta &&\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &\\
0 & -\frac{13ml}{420} & 0 & \frac{(l)^3}{30}Q_\zeta & 0 & -\frac{m(l)^2}{140} & 0 & -\frac{11ml}{210} & 0 & \frac{(l)^3}{20}Q_\zeta & 0 & \frac{m(l)^2}{105}
\end{bmatrix}^{ij}$$

$$\left[\int_V \rho S_3^T S_3 dV\right]^{ij} =$$

$$=\begin{bmatrix}
0 &&&&&&&&&&&\\
0 & 0 &&&&&&&&&&\\
0 & 0 & \frac{13m}{55} &&&&&& \textit{symmetric} &&&\\
0 & 0 & \frac{7(l)^2}{20}Q_\eta & \frac{(l)^3}{3}I_\zeta &&&&&&&&\\
0 & 0 & -\frac{11ml}{210} & -\frac{(l)^3}{20}Q_\eta & \frac{m(l)^2}{105} &&&&&&&\\
0 & 0 & 0 & 0 & 0 & 0 &&&&&&\\
0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&\\
0 & 0 & \frac{9m}{70} & \frac{3(l)^2}{20}Q_\eta & -\frac{13ml}{420} & 0 & 0 & 0 & \frac{13m}{35} &&&\\
0 & 0 & \frac{3(l)^2}{20}Q_\eta & \frac{(l)^3}{6}I_\eta & -\frac{(l)^3}{30}Q_\eta & 0 & 0 & 0 & \frac{7(l)^2}{20}Q_\eta & \frac{(l)^3}{3}I_\zeta &&\\
0 & 0 & \frac{13m}{420} & \frac{(l)^3}{30}Q_\eta & -\frac{m(l)^2}{140} & 0 & 0 & 0 & \frac{11ml}{210} & \frac{(l)^3}{20}Q_\eta & \frac{m(l)^2}{105} &\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^{ij}$$

$$\left[\int_V \rho S_1^T S_2 dV\right]^{ij} =$$

$$= \begin{bmatrix}
0 & \frac{7m}{20} & 0 & -\frac{(l)^2}{3}Q_\zeta & 0 & \frac{ml}{20} & 0 & \frac{3m}{20} & 0 & -\frac{(l)^2}{6}Q_\zeta & 0 & -\frac{ml}{30} \\
0 & \frac{l}{2}Q_\eta & 0 & -\frac{(l)^2}{2}I_{\eta\zeta} & 0 & \frac{(l)^2}{10}Q_\eta & 0 & \frac{l}{2}Q_\eta & 0 & -\frac{(l)^2}{2}I_{\eta\zeta} & 0 & -\frac{(l)^2}{10}Q_\eta \\
0 & \frac{l}{2}Q_\zeta & 0 & -\frac{(l)^2}{2}I_\eta & 0 & \frac{(l)^2}{10}Q_\zeta & 0 & \frac{l}{2}Q_\zeta & 0 & -\frac{(l)^2}{2}I_\eta & 0 & -\frac{(l)^2}{10}Q_\zeta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(l)^2}{10}Q_\zeta & 0 & -\frac{(l)^3}{12}I_\eta & 0 & 0 & 0 & -\frac{(l)^2}{10}Q_\zeta & 0 & \frac{(l)^3}{12}I_\eta & 0 & \frac{(l)^3}{60}Q_\zeta \\
0 & -\frac{(l)^2}{10}Q_\eta & 0 & \frac{(l)^3}{12}I_{\eta\zeta} & 0 & 0 & 0 & \frac{(l)^2}{10}Q_\eta & 0 & -\frac{(l)^3}{12}I_{\eta\zeta} & 0 & -\frac{(l)^3}{60}Q_\eta \\
0 & \frac{3m}{20} & 0 & -\frac{(l)^2}{6}Q_\zeta & 0 & \frac{ml}{30} & 0 & \frac{7m}{20} & 0 & -\frac{(l)^2}{3}Q_\zeta & 0 & -\frac{ml}{20} \\
0 & -\frac{l}{2}Q_\eta & 0 & \frac{(l)^2}{2}I_{\eta\zeta} & 0 & -\frac{(l)^2}{10}Q_\eta & 0 & -\frac{l}{2}Q_\eta & 0 & \frac{(l)^2}{2}I_{\eta\zeta} & 0 & \frac{(l)^2}{10}Q_\eta \\
0 & -\frac{l}{2}Q_\zeta & 0 & \frac{(l)^2}{2}I_\eta & 0 & -\frac{(l)^2}{10}Q_\zeta & 0 & -\frac{l}{2}Q_\zeta & 0 & \frac{(l)^2}{2}I_\eta & 0 & \frac{(l)^2}{10}Q_\zeta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{(l)^2}{10}Q_\zeta & 0 & \frac{(l)^3}{12}I_\eta & 0 & -\frac{(l)^3}{60}Q_\zeta & 0 & \frac{(l)^2}{10}Q_\zeta & 0 & -\frac{(l)^3}{12}I_\eta & 0 & 0 \\
0 & \frac{(l)^2}{10}Q_\eta & 0 & -\frac{(l)^3}{12}I_{\eta\zeta} & 0 & \frac{(l)^3}{60}Q_\eta & 0 & -\frac{(l)^2}{10}Q_\eta & 0 & \frac{(l)^3}{12}I_{\eta\zeta} & 0 & 0
\end{bmatrix}^{ij}$$

$$\left[\int_V \rho S_1^T S_3 dV\right]^{ij} =$$

$$= \begin{bmatrix}
0 & 0 & \frac{7m}{20} & \frac{(l)^2}{3}Q_\eta & -\frac{ml}{20} & 0 & 0 & 0 & \frac{3m}{20} & \frac{(l)^2}{6}Q_\eta & -\frac{ml}{30} & 0 \\
0 & 0 & \frac{l}{2}Q_\eta & \frac{(l)^2}{2}I_\zeta & -\frac{(l)^2}{10}Q_\eta & 0 & 0 & 0 & \frac{l}{2}Q_\eta & \frac{(l)^2}{2}I_\zeta & \frac{(l)^2}{10}Q_\eta & 0 \\
0 & 0 & \frac{l}{2}Q_\zeta & \frac{(l)^2}{2}I_{\eta\zeta} & -\frac{(l)^2}{10}Q_\zeta & 0 & 0 & 0 & \frac{l}{2}Q_\zeta & \frac{(l)^2}{2}I_{\eta\zeta} & \frac{(l)^2}{10}Q_\zeta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{(l)^2}{10}Q_\zeta & \frac{(l)^3}{12}I_{\zeta\eta} & 0 & 0 & 0 & 0 & -\frac{(l)^2}{10}Q_\zeta & -\frac{(l)^3}{12}I_{\eta\zeta} & -\frac{(l)^2}{60}Q_\zeta & 0 \\
0 & 0 & -\frac{(l)^2}{10}Q_\eta & -\frac{(l)^3}{12}I_\zeta & 0 & 0 & 0 & 0 & \frac{(l)^2}{10}Q_\eta & \frac{(l)^3}{12}I_\zeta & \frac{(l)^3}{60}Q_\eta & 0 \\
0 & 0 & \frac{3m}{20} & \frac{(l)^2}{6}Q_\eta & -\frac{ml}{30} & 0 & 0 & 0 & \frac{7m}{20} & \frac{(l)^2}{3}Q_\eta & \frac{ml}{20} & 0 \\
0 & 0 & -\frac{l}{2}Q_\eta & -\frac{(l)^2}{2}I_\zeta & \frac{(l)^2}{10}Q_\eta & 0 & 0 & 0 & -\frac{l}{2}Q_\eta & -\frac{(l)^2}{2}I_\zeta & -\frac{(l)^2}{10}Q_\eta & 0 \\
0 & 0 & -\frac{l}{2}Q_\zeta & -\frac{(l)^2}{2}I_{\eta\zeta} & \frac{(l)^2}{10}Q_\zeta & 0 & 0 & 0 & -\frac{l}{2}Q_\zeta & -\frac{(l)^2}{2}I_{\eta\zeta} & -\frac{(l)^2}{10}Q_\zeta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{(l)^2}{10}Q_\zeta & -\frac{(l)^3}{12}I_{\eta\zeta} & \frac{(l)^3}{60}Q_\zeta & 0 & 0 & 0 & \frac{(l)^2}{10}Q_\zeta & \frac{(l)^3}{12}I_{\eta\zeta} & 0 & 0 \\
0 & 0 & \frac{(l)^2}{10}Q_\eta & \frac{(l)^3}{12}I_\zeta & -\frac{(l)^3}{60}Q_\eta & 0 & 0 & 0 & \frac{(l)^2}{10}Q_\eta & -\frac{(l)^3}{12}I_\zeta & 0 & 0
\end{bmatrix}^{ij}$$

$$\left[\int_V \rho S_2^T S_3 dV\right]^{ij} =$$

$$= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{13m}{35} & \frac{7(l)^2}{20}Q_\eta & -\frac{11ml}{210} & 0 & 0 & 0 & \frac{9m}{70} & \frac{3(l)^2}{20}Q_\eta & \frac{13ml}{420} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{7(l)^2}{20}Q_\zeta & -\frac{(l)^3}{3}I_{\eta\zeta} & \frac{(l)^3}{20}Q_\zeta & 0 & 0 & 0 & -\frac{3(l)^2}{20}Q_\zeta & -\frac{(l)^3}{6}I_{\eta\zeta} & -\frac{(l)^3}{30}Q_\zeta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{11ml}{210} & \frac{(l)^3}{20}Q_\eta & -\frac{m(l)^2}{105} & 0 & 0 & 0 & \frac{13ml}{420} & \frac{(l)^3}{30}Q_\eta & \frac{m(l)^2}{140} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{9m}{70} & \frac{3(l)^2}{20}Q_\eta & -\frac{13ml}{420} & 0 & 0 & 0 & \frac{13m}{35} & \frac{7(l)^2}{20}Q_\eta & \frac{11ml}{210} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3(l)^2}{20}Q_\zeta & -\frac{(l)^3}{6}I_{\eta\zeta} & \frac{(l)^3}{30}Q_\zeta & 0 & 0 & 0 & -\frac{7(l)^2}{20}Q_\zeta & -\frac{(l)^3}{3}I_{\eta\zeta} & \frac{(l)^3}{20}Q_\zeta & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{13ml}{420} & -\frac{(l)^3}{30}Q_\eta & \frac{m(l)^2}{140} & 0 & 0 & 0 & -\frac{11ml}{210} & -\frac{(l)^2}{20}Q_\eta & -\frac{m(l)^2}{105} & 0
\end{bmatrix}^{ij}$$

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