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A Singularity Avoiding Track Model for Race Car Optimization

Adwait Verulkar, Behzad Danaei, Srikanth Sivaramakrishnan, Ryan Trinter
General Motors

Agenda

- Formula 1 context: why the track model matters
- Motivation: the Frenet singularity
- Fitting pipeline
- Modelica TrackJoint implementation
- Results: Singapore minimum lap time
- Conclusions

Why the Track Model Matters for F1 Lap-Time Optimization

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Setup and energy decisions are made in simulation, on a model of every track.

1. **Every circuit, before the car gets there**

   Few test days and a cost cap: all 23 circuits are simulated, so each track model has to come straight from a LiDAR scan.

2. **The “perfect driver” needs a real road**

   Minimum-lap-time optimal control finds the fastest lap; camber, crown and elevation change the loads it can use.

3. **Tight corners decide lap time**

   Monaco and Singapore hairpins are tighter than the road is wide, exactly where centre-line track models break down.

4. **One model, not two**

   The production Modelica car drives the simulators; the track must plug in as a standard MultiBody component.

USE CASE 1 · SETUP PER CIRCUIT

Which car for which track?

- Ride heights and aero balance respond to bank, grade and bumps
- Lap-time sensitivities need a road that is smooth to differentiate

USE CASE 2 · ENERGY MANAGEMENT

Where to deploy, where to harvest?

- Deployment on corner exits depends on corner geometry
- Speed along the track sets the time–distance transform

Goal: optimize the production Modelica car on a measured 3D track

Vehicle + TrackJoint exported once from Dymola as FMI 3.0 Model Exchange with LS-DAE; run on all 23 circuits.

Minimum-Lap-Time Optimal Control Problem

Posed in distance s: the track supplies ṡ, the bounds on w and the road surface.

$$\min_{x(\cdot),\,z(\cdot),\,u(\cdot)} \quad t(L) \;+\; \varepsilon\, R(u)$$

lap time + small control regularization

$$\text{s.t.} \quad \frac{dx}{ds} = \frac{1}{\dot{s}(x)}\, f(x, z, u, p)$$

vehicle dynamics from the FMU, s = track distance

$$0 = g(x, z, u, p)$$

LS-DAE residuals: closed suspension loops

$$w_R(s) \le w(s) \le w_L(s)$$

track limits

$$h(x, z, u, p) \le 0$$

tyre saturation, MGU-K power & speed derate

$$0 \le \mathrm{SoC}(s) \le 1, \quad E_{\mathrm{harv}}(L) \le 1$$

battery window 4 MJ, harvest 8.5 MJ / lap

$$0 \le a(s) \le a_{\max}(s)$$

active-aero zones

$$x(0) \approx x(L)$$

periodic, flying lap

$$u_{\min} \le u \le u_{\max}, \quad p = \bar{p}$$

actuator limits; setup offsets fixed

TRANSCRIPTION (FMIOPT + CASADI)

- Direct collocation in s, IPOPT + MUMPS
- FMU Jacobians via FMI 3 directional derivatives
- LS-DAE: algebraic variables and residuals become NLP variables and constraints
- Singapore: converges only with the minimum-curvature reference

WHERE THE TRACK ENTERS

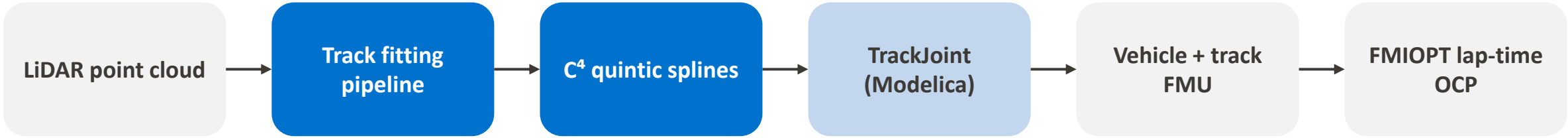
- ṡ = vsTrack, exported by TrackJoint, rescales every derivative
- Track limits w_R(s) ≤ w ≤ w_L(s) from the fitted width splines
- Height, bank and grade act through the 3D surface

Conditioning hinges on the factor 1 − κw inside ṡ

A centre-line reference drives it to zero in tight corners; the minimum-curvature line keeps it away from zero.

From LiDAR Scan to Lap Time

The same chain runs for every circuit on the 2026 F1 calendar; these two papers build the track part of it.



Input

- Surveyed LiDAR scan of each circuit: x, y, z and surface normals
- 23 circuits, from Monaco (3.3 km) to Spa (6.9 km)

Output for the team

- Optimal racing line, speed, aero mode and energy use from one solve
- Setup comparisons and lap-time sensitivities per circuit

Without a usable 3D track, none of this runs

The track sets the time-to-distance transform, the track limits and the road loads in every lap.

What an F1 Track Model Must Deliver

Requirement	Why it matters in Formula 1
Accurate 3D surface	Camber, crown and grade change loads and grip; Zandvoort banks up to 16.2°
Regular in tight corners	Monaco and Singapore have corners tighter than the road is wide
Smooth (C^4)	The DAE solver and the NLP differentiate the road at every collocation point
Reusable	One compiled FMU must run all 23 circuits without rebuilding
Cheap to evaluate	About two thirds of a 464 s Singapore solve is spent inside the FMU

This paper: a track model that stays well defined in every corner

A fitting pipeline anchored on the minimum-curvature line, exported as C^4 splines.

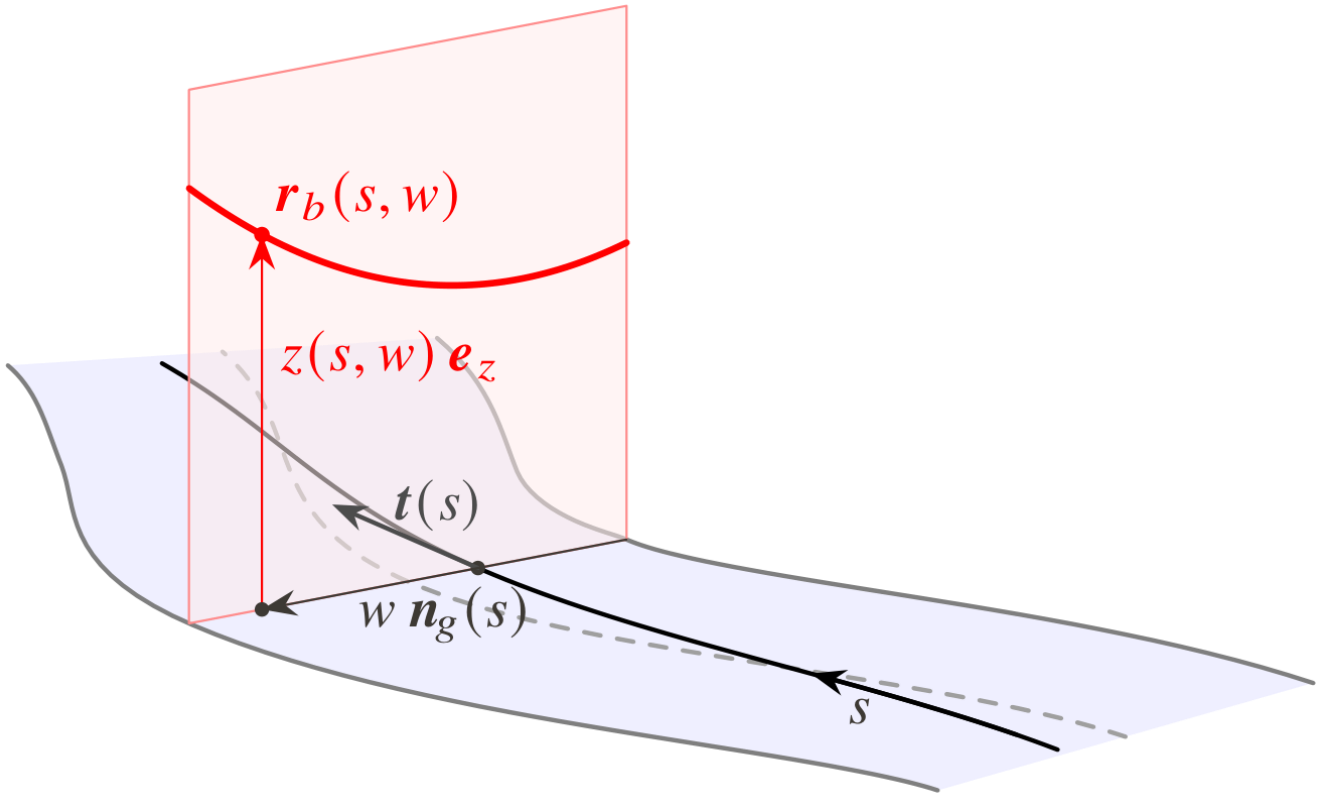
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Motivation

The Track Model Matters

- Normal loads, gravity projections and wheel constraints are all expressed through the **road surface and its first two derivatives**
- Track-model errors propagate into lap time, tyre forces, suspension loads and aero
- Curvilinear (s, n) road description since Cossalter et al. (1999)
  - Frenet–Serret ribbon (Perantoni and Limebeer, 2015): assumes lateral flatness
  - Ribbon + Monge elevation correction (Lovato 2021; Limebeer 2023)
  - Free Monge patch (Limebeer and van de Merwe, 2025): no Euler angles, closed-form fundamental forms
- **Open question**: which reference line should anchor the (s, w) frame?

Monge-Patch Track Surface



Grey: track boundaries; dashed: geometric centre-line; black: minimum-curvature reference line r(s); red: lateral height polynomial

$$r_b(s,w) = r(s) + w\,n_g(s) + z(s,w)\,e_z$$

$$z(s,w) = \sum_{k=0}^{N} z_k(s)\,w^k, \qquad N = 3$$

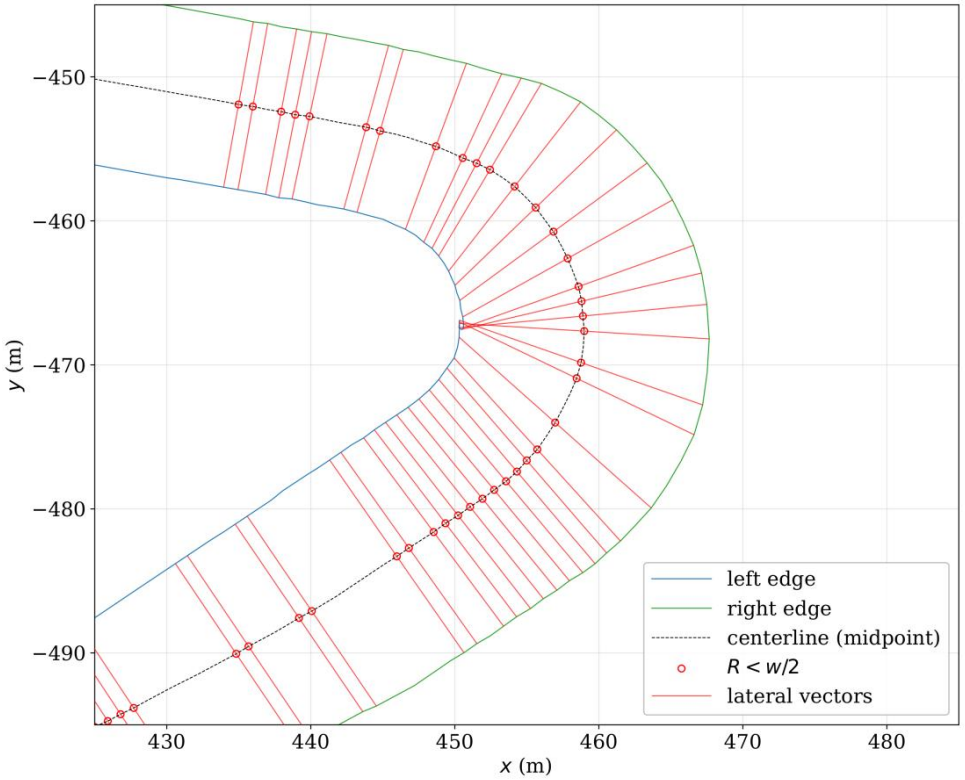
- t(s): unit tangent; n_g(s): unit left normal of the reference line
- Low order **N = 3** captures camber and crown on every circuit processed
- Camber angle ϕ = arctan(∂_w z)

The Frenet Singularity

$$\partial_s \boldsymbol{r}_b = \big(1 - \kappa_c(s)\, w\big)\boldsymbol{t}(s) + \partial_s z\, \boldsymbol{e}_z$$
$$\partial_w \boldsymbol{r}_b = \boldsymbol{n}_g(s) + \partial_w z\, \boldsymbol{e}_z$$

$$\sqrt{\det I_P} = \left\|\partial_s \boldsymbol{r}_b \times \partial_w \boldsymbol{r}_b\right\| \approx \left|1 - \kappa_c(s)\, w\right| \sqrt{1 + \|\nabla z\|^2}$$
$$w_{\text{fold}}(s) = \frac{1}{\kappa_c(s)}$$

- Jacobian vanishes at the **centre of curvature** of the reference line
- Centre-line radius below the half-width puts the fold **inside the drivable region**
- Neighbouring laterals cross: one 3D point gets several (s, w) addresses; tyre-load equations become ill-posed
- Observed repeatedly on hairpins and slow sequences, e.g. **Singapore**



Why It Hurts Minimum Lap-Time Problems

$$\dot{s} = \frac{v_x \cos\chi - v_y \sin\chi}{1 - \kappa_c(s)\, w}$$

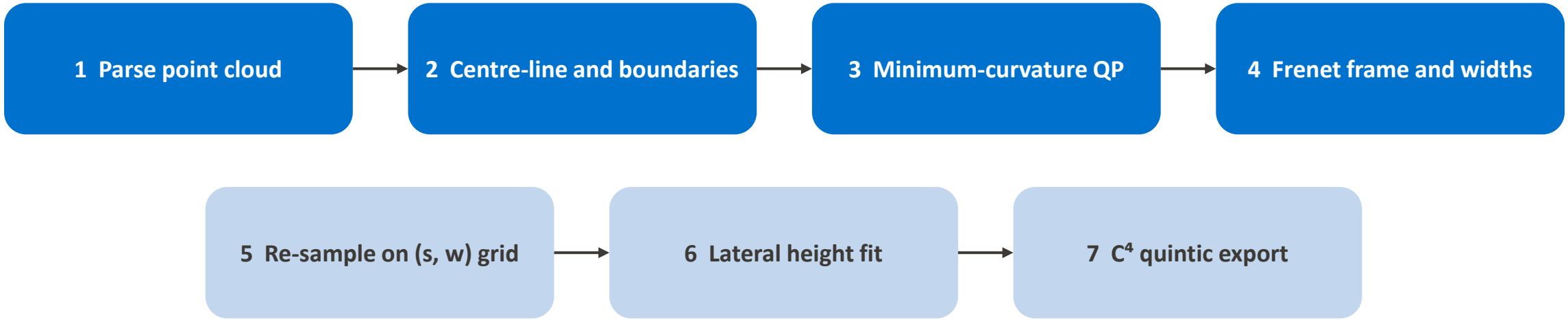
- Spatial-domain transcription scales vehicle and track dynamics by 1/ṡ
- Denominator 1 − κ_c w is the **same Jacobian factor**: as w → w_fold, ṡ → ∞ and 1/ṡ → 0
  - Stiff, ill-conditioned NLP with a singular Hessian **exactly at the corners that matter**
- Secondary effect: in cambered slow corners the largest height gradients sit at large |w|
  - High-order lateral coefficients grow; spurious wiggles in ∂_w z, ∂_ww z drive the tyre normal load
- Previous workaround: add a small lateral offset to the centre-line at the offending stations

Proposed Solution: Minimum-Curvature Reference Line

- Anchor the Monge patch on the **minimum-curvature line (MCL)** (Braghin 2008; Heilmeier 2020)
  - Used as the curvilinear reference, **not as a driving target**
- Lower curvature peaks move the fold **outside the drivable strip**
- MCL hugs the inside boundary in slow corners: admissible |w| becomes one-sided and smaller
  - Better conditioning of the lateral fit and of the 1/ṡ transformation
- Pipeline: point cloud → **C^4 periodic piecewise-quintic splines** → Modelica TrackJoint
- Only requirement: point cloud with unique, sorted longitudinal and lateral indices

Fitting Pipeline

Pipeline Overview



- Structured surveyor cloud (j, k, x, y, z, n): j longitudinal station, k lateral sample
- QP iterated on its linearisation, with periodic α-smoothing against active-constraint chatter
- Widths b_left(s), b_right(s) by ray–polyline intersection with the boundaries
- Channels r_x, r_y, z_0 ... z_N, b_left, b_right exported on one shared knot vector
- Roughly **5 minutes per circuit**; reproduces surveyed elevations and bounds to a few centimetres

Minimum-Curvature Quadratic Program

$$\boldsymbol{r}_i = \boldsymbol{p}_i + \alpha_i \boldsymbol{n}_i, \qquad \alpha_i \in \left[-b_{\mathrm{r},i} + \tfrac{b_v}{2},\ b_{\mathrm{l},i} - \tfrac{b_v}{2}\right]$$

$$\min_{\boldsymbol{\alpha}} \tfrac{1}{2}\boldsymbol{\alpha}^\top H \boldsymbol{\alpha} + f^\top \boldsymbol{\alpha}, \qquad H = A^\top A, \quad f = -A^\top \boldsymbol{c}$$

$$\underbrace{\textstyle\sum_i \kappa_i^2}_{\text{curvature}} + \underbrace{w_L \textstyle\sum_i \|\boldsymbol{r}_{i+1} - \boldsymbol{r}_i\|^2}_{\text{length}} + \underbrace{w_J \textstyle\sum_i \|\boldsymbol{r}_{i+2} - 3\boldsymbol{r}_{i+1} + 3\boldsymbol{r}_i - \boldsymbol{r}_{i-1}\|^2}_{\text{jerk}}$$

- Candidate points move along the reference normal; bounds shrunk by half the vehicle width b_v
- Curvature as second finite differences; **length** cost against stretching, **jerk** cost against two-sample chatter at active bounds
- Rows weighted by Δs^(−3/2) and Δs^(−5/2) for non-uniform sampling
- Periodic circulant difference operators give a **banded, positive semidefinite** Hessian; solved with **OSQP**

Iteration, Containment and Smoothing

$$\boldsymbol{p}_i^{(k+1)} \leftarrow \boldsymbol{p}_i^{(k)} + \alpha_i^{(k)}\, \boldsymbol{n}_i^{(k)}$$

One QP is a frozen-reference step

- Curvature linearised about the current line
- Stale arc length, normals and stencil weights
- Stale ray-cast lateral bounds

Fixed-point outer loop (SQP structure)

- Re-sample, recompute normals, Δs and widths; **3–5 iterations**

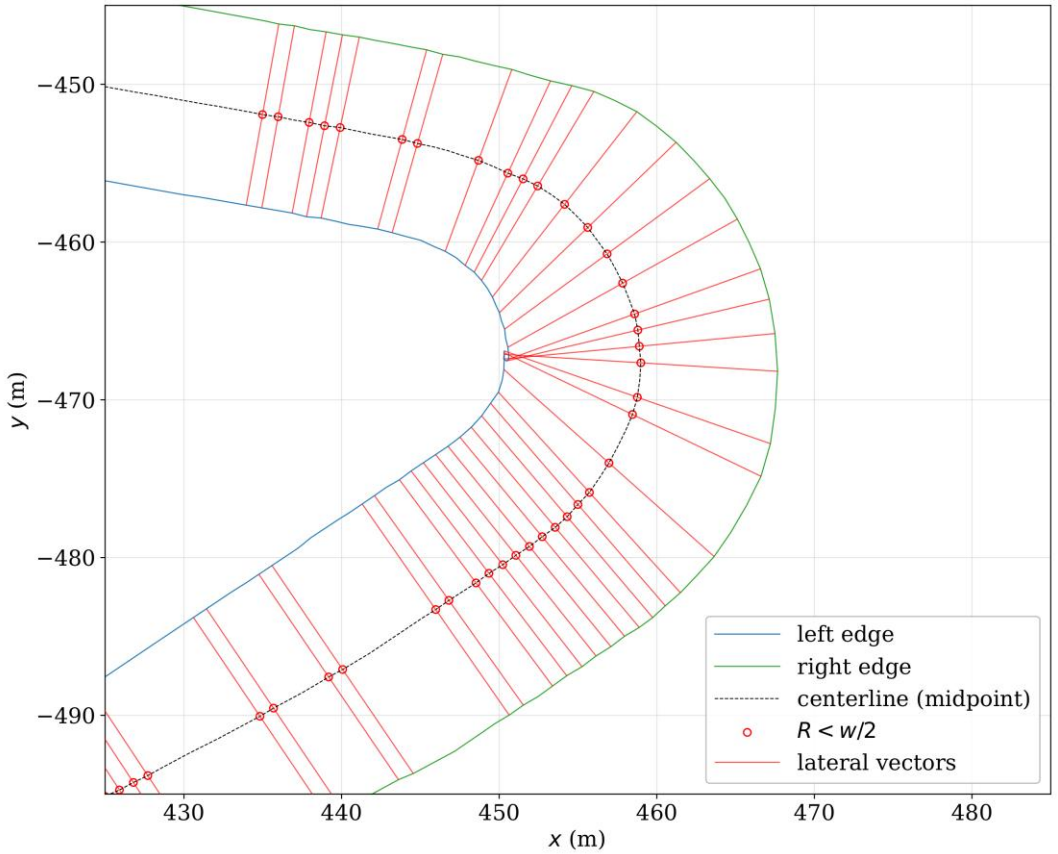
Why not a general NLP solver?

- Ray–polyline widths, polygon containment and arc-length resampling are **non-smooth** in α
- Frozen, each subproblem is a convex QP with banded Hessian H = A^TA

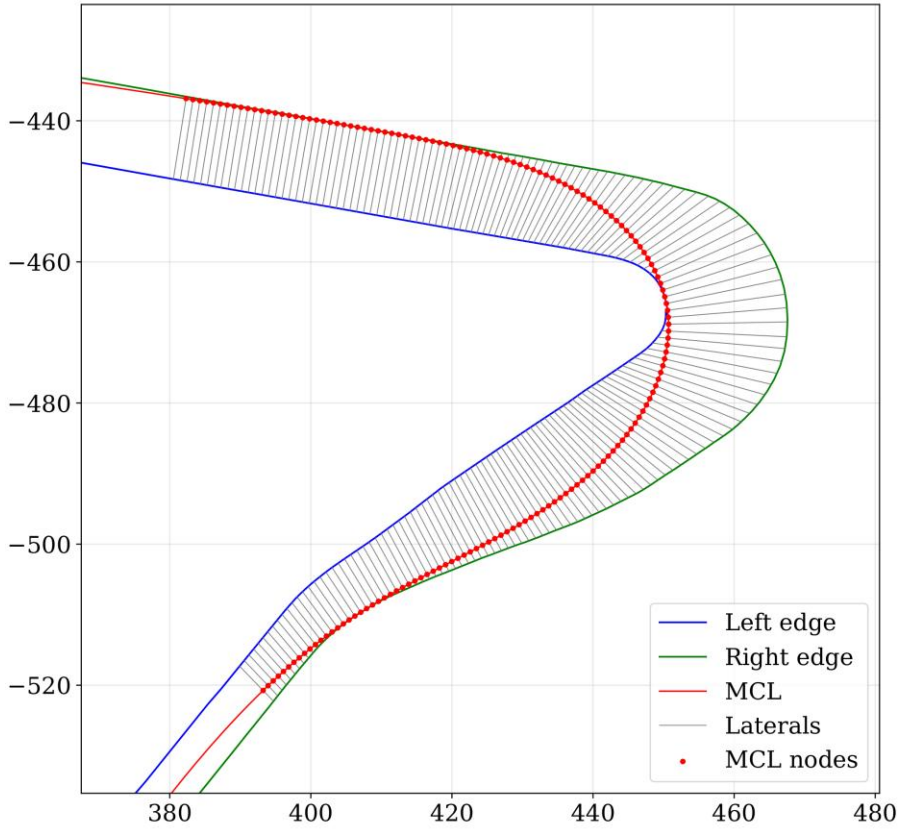
Containment and smoothing

- Inside/outside test (contains_points) after every solve; reject and revert to the last fully contained iterate
- Short periodic centred moving average on α removes zig-zag at active bounds

Result: Laterals No Longer Cross

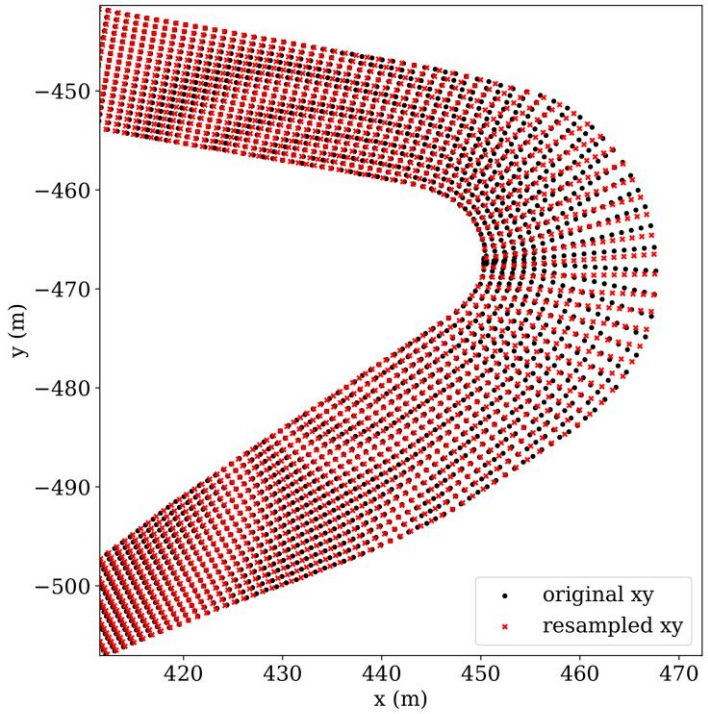


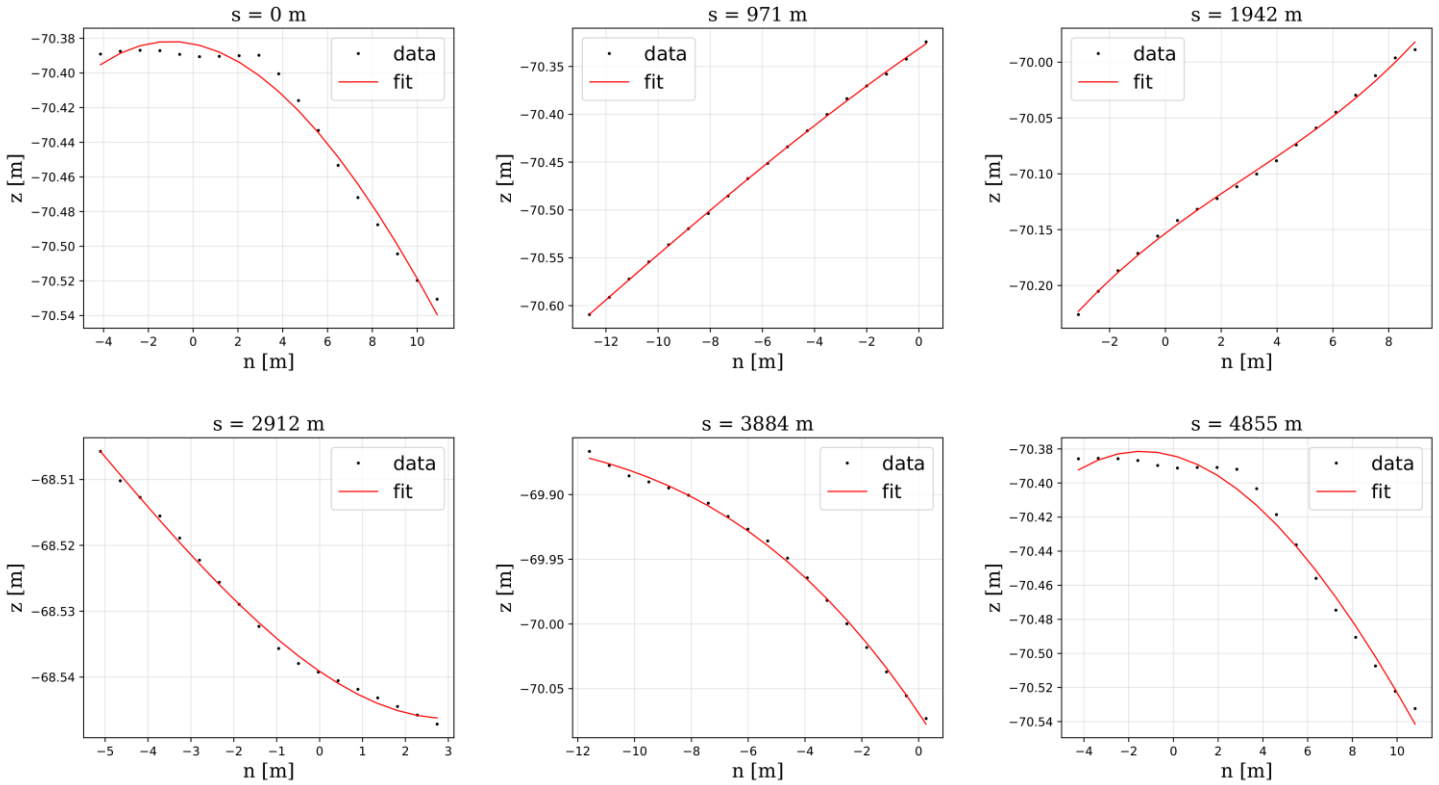
Geometric centre-line: laterals intersect inside the road



Minimum-curvature line: laterals stay injective

Re-sampling and Monge-Patch Fit





- Regular grid **orthonormal to the MCL**; lateral stations sweep w ∈ [−b_right, b_left]
- Elevation queried from a high-density surface: Delaunay interpolation, thin-plate-spline RBF fallback outside the hull
- Per-station least-squares fit of the degree-N lateral polynomial

Export: C^4 Periodic Piecewise Quintic

$$y_i(s) = \sum_{j=0}^{5} a_{i,j}\,(s - s_i)^j, \quad s \in [s_i, s_{i+1}]$$

$$\min_{a} \frac{1}{m} \sum_{p=1}^{m} \left(y_{i(p)}(s_p) - y_p\right)^2 + \sum_{q=1}^{4} w_q \int_0^L \left(y^{(q)}(s)\right)^2 \mathrm{d}s \quad \text{s.t. } C^4 \text{ at knots and wrap}$$

- N + 5 channels on **one shared knot sequence**; value and first four derivatives periodic at the wrap
- CasADi NLP solved by IPOPT with HSL MA57; derivative penalties by 3-point Gauss–Legendre quadrature
- Zero-phase FFT low-pass on elevation channels (cut-off = segment length) avoids under-resolved oscillations
- **Why quintic**: the joint needs third s-derivatives, index reduction adds one more, so C^4 is the minimum

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Modelica Implementation

TrackJoint Consumes the Splines

Parameter record

```
parameter Real sBreak[nSeg + 1];
parameter Real rxCoeff[nSeg, d + 1];
parameter Real ryCoeff[nSeg, d + 1];
parameter Real rzCoeff[nSeg, d + 1];
parameter Real z1Coeff[nSeg, d + 1];
parameter Real z2Coeff[nSeg, d + 1];
parameter Real z3Coeff[nSeg, d + 1];
```

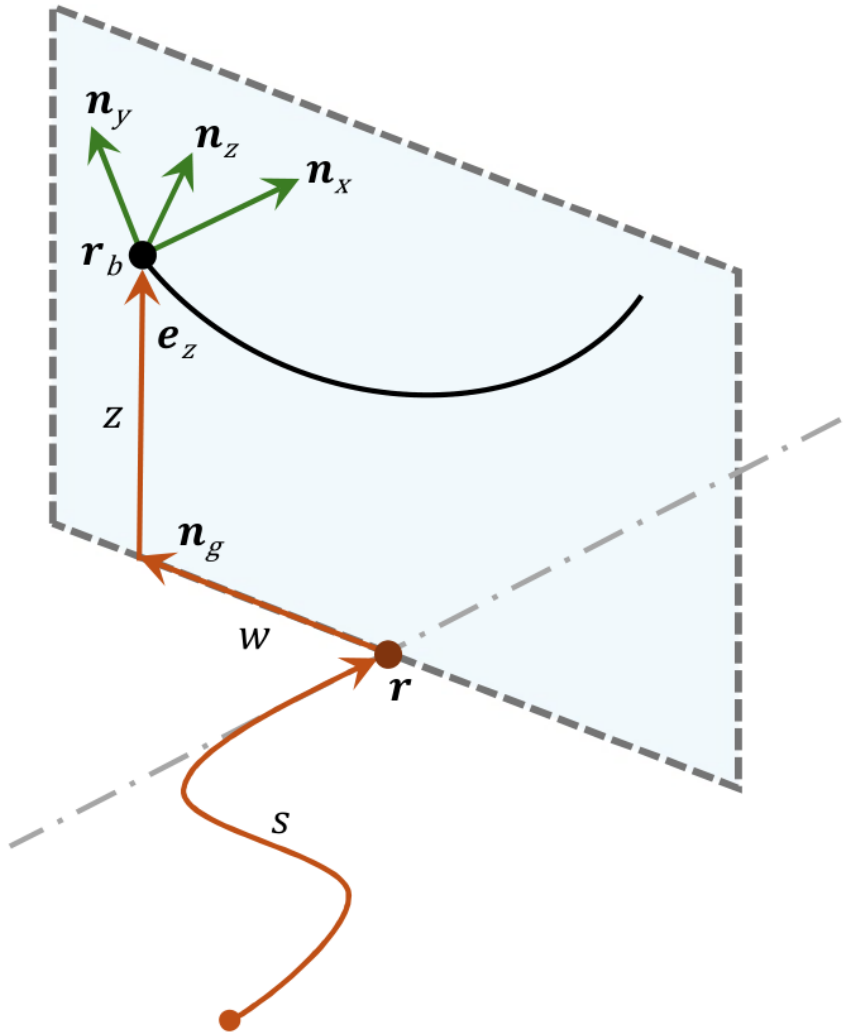
$$y^{(q)}(s) = \sum_{j=q}^{d} \frac{j!}{(j-q)!}\, a_{\mathrm{ix},j}\, (s - s_{\mathrm{ix}})^{\,j-q}, \qquad \mathrm{ix} = \min\left(\left\lfloor \frac{s \bmod L}{\Delta s} \right\rfloor,\ N_{\mathrm{seg}} - 1\right)$$

- Joint uses the geometry channels {r_x, r_y, r_z, z_1, z_2, z_3}
- Widths become the OCP path constraint −b_right(s) ≤ w ≤ b_left(s)
- Shared breaks: segment index found **once per step**, reused by all channels
- Periodic export: no special case at start/finish; s reduced modulo L
- Constant-time lookup and Horner evaluation are **branch-free**: smooth expression graph for CasADi
- Coefficients are **tunable**: retarget a compiled FMU to a new circuit

Constraint, Assembly and FMI 3.0 Export

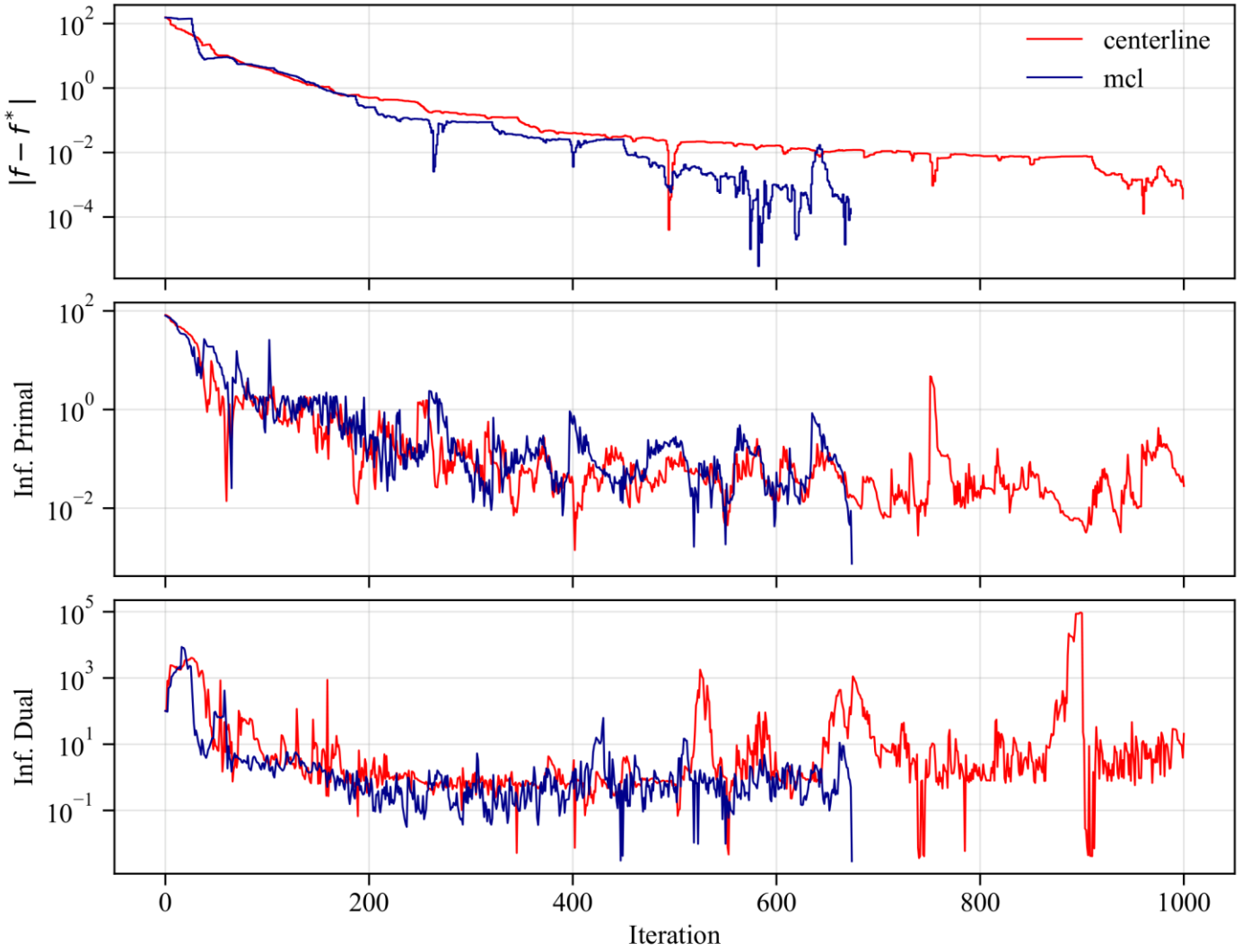
$$0 = \boldsymbol{f}_b^w \cdot \partial_s \boldsymbol{r}_b + \boldsymbol{\tau}_b^w \cdot \partial_s \boldsymbol{\phi}$$
$$0 = \boldsymbol{f}_b^w \cdot \partial_w \boldsymbol{r}_b + \boldsymbol{\tau}_b^w \cdot \partial_w \boldsymbol{\phi}$$

- Triad built analytically: **no Euler angles or quaternions** integrated in the joint
- Two d'Alembert relations close the index-1 DAE; their 2×6 Jacobian **loses rank at the fold**
  - The MCL keeps the Modelica constraint solvable, not only the NLP well conditioned
- Heave prismatic along n_z + roll/pitch/yaw revolutes **relative to the track triad**
- One FMI 3.0 model-exchange FMU; **vsTrack = ṡ** drives the time-to-space transformation
- Track-relative coordinates: zero chassis states + linear s-ramp is a feasible full-lap guess



Results

IPOPT Convergence: Singapore



- Same F1 multibody vehicle and solver settings; only the track reference changes

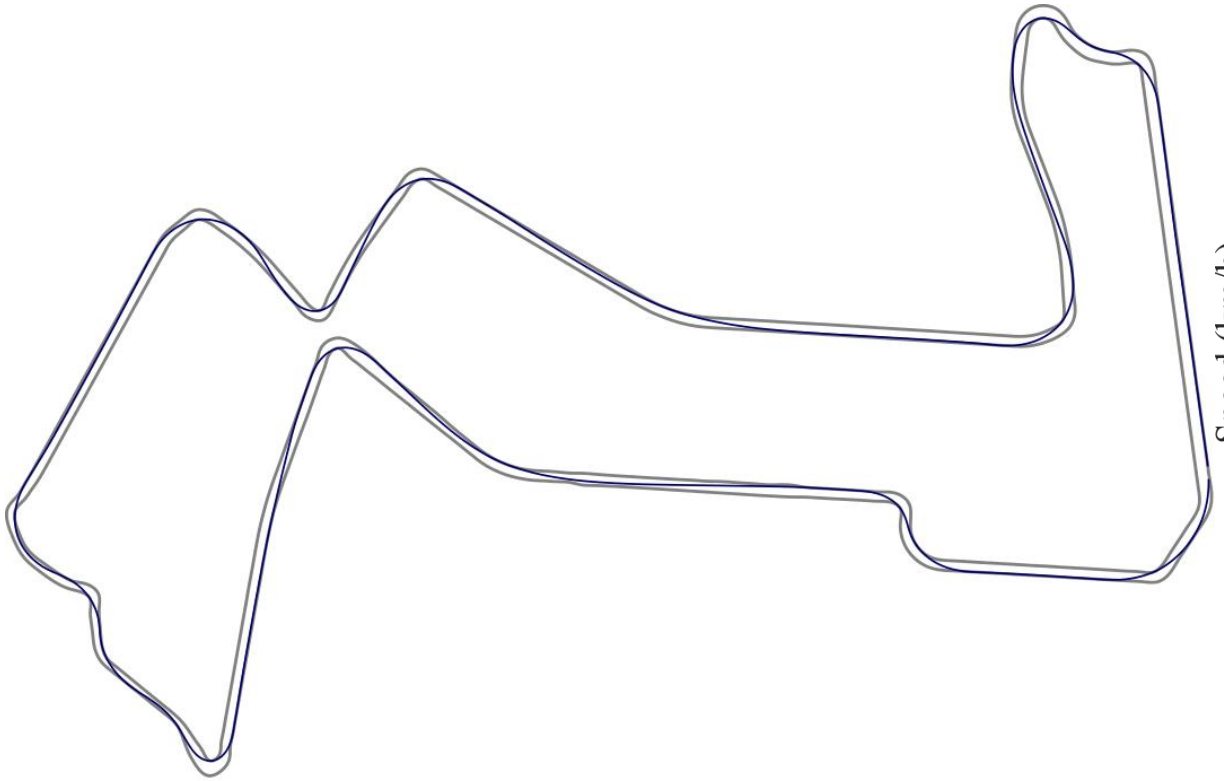
Centre-line

- Ribbon + Monge patch (Limebeer 2023)
- Dual infeasibility jumps by orders of magnitude; primal residual stalls; IPOPT adds regularisation
- Does not converge

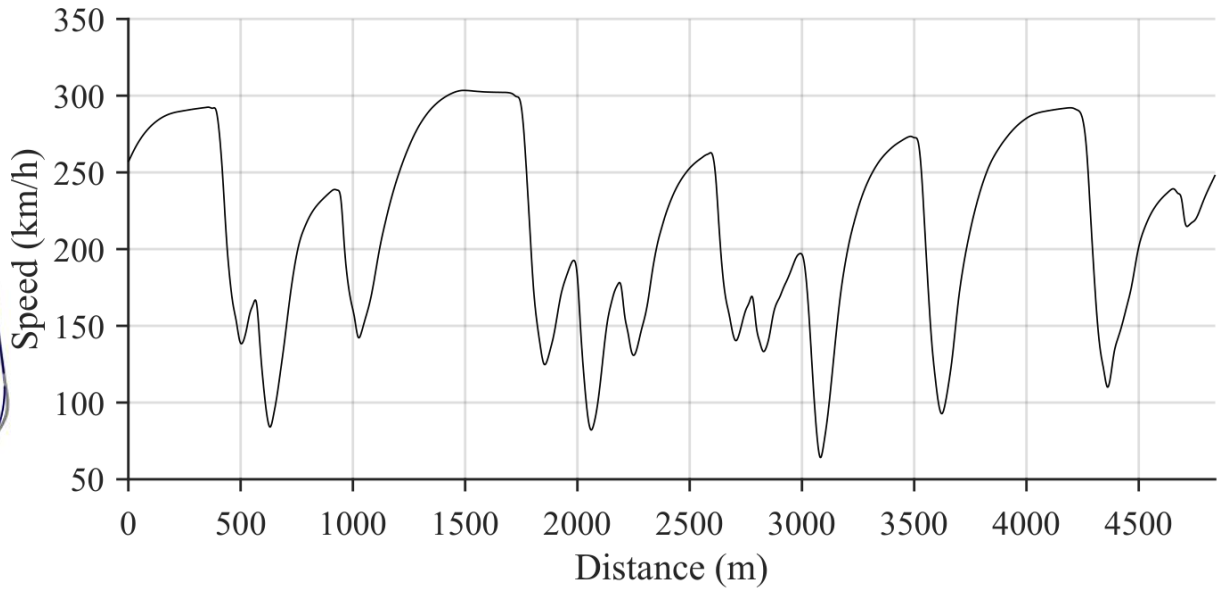
Minimum-curvature line

- Fold moved outside the drivable strip
- Converges to IPOPT tolerances
- Previously intractable cases solved **without changing the vehicle model or solver settings**

Optimal Solution: Singapore



Optimal racing line within the drivable strip



Optimal speed along the reference arc length s

Conclusions

Conclusions

- A centre-line reference is **unsafe in any corner tighter than the road is wide**
- Anchoring the Monge patch on the **minimum-curvature line** removes the Frenet singularity and improves NLP conditioning
- Pipeline: measured point cloud → closed, **C^4 periodic quintic** surface in about 5 minutes per circuit
- Export format (shared knots, quintic, periodic) is set by what the Modelica **TrackJoint** differentiates
- Plug-in track interface for the FMIOPT optimal-control toolchain

Future work

- Adaptive mesh instead of a rectangular (s, n) grid
- Dynamic racing line as reference for a more linear time transformation

Appendix

Appendix: Circuit Coverage, 2026 F1 Calendar

Circuit	L [m]	N_seg	R_min [m]	Bank [°]	Margin
Monaco	3283	657	9.3	5.2	1.03
Zandvoort	4185	837	31.2	16.2	2.32
Singapore	4858	972	23.0	2.1	1.56
Spa	6934	1387	22.5	4.6	1.42
Silverstone	5798	1160	41.2	1.9	2.38
All 23 (range)	3283–6934	657–1387	9.3–41.2	1.5–16.2	**1.03**–2.89

Margin = min over s of 1 / (|κ| b)

- 23 circuits, uniform **5 m knot spacing**
- 657–1387 quintic segments; ≈ 1.2 MB JSON per circuit
- Injectivity margin **> 1 on all 23** circuits with the minimum-curvature reference
- Tightest: **Monaco 1.03** (9.3 m reference-line radius)
- Steepest camber: Zandvoort, 16.2° bank


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